Resummation of transverse momentum distributions in distribution space.

Markus Ebert
Deutsches Elektronen-Synchrotron

In collaboration with Frank Tackmann

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PIER

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DESY

Motivation.

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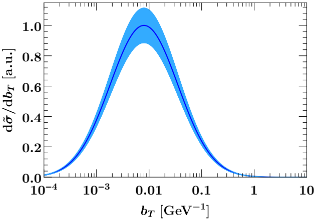
- Transverse momenta $q_T \ll m_H$ are dominated by Sudakov logarithms $\alpha_s^n \ln^m(q_T/m_H)\,,\ m < 2n \to$ Resummation to all orders is necessary
- Resummation is well understood since [Collins, Soper, Sterman '85]
- Resummation carried out mostly in Fourier space $\vec{b}_T$

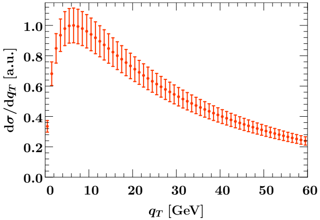
  DYRes [Catani, de Florian, Ferrera, Grazzini '15 ...], HRes [de Florian, Ferrera, Grazzini, Tommasini '12 ...], ResBos [Wang, Li[3], Yuan '12 ...], CuTe [Becher, Neubert, Wilhelm '12], [D'Alesio, Echevarria, Melis, Scimemi '14], [Echevarria, Kasemets, Mulders, Pisano '15], [Neill, Rothstein, Vaidya '15], arTeMiDe [Scimemi, Vladimirov '17], . . .

  - Resums $\ln(Qb_T)$ rather than $\ln(Q/q_T)$
  - Theory uncertainties are estimated in Fourier space:

    Scale variations probe $\ln(Qb_T)$ rather than $\ln(Q/q_T)$



$\xrightarrow{\textbf{FT}}$



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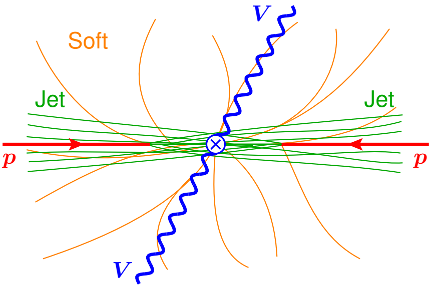
  - Resums $\ln(Qb_T)$ rather than $\ln(Q/q_T)$
  - Theory uncertainties are estimated in Fourier space:
    Scale variations probe $\ln(Qb_T)$ rather than $\ln(Q/q_T)$
- Is it possible to carry out resummation directly in momentum space?
  [Frixione, Nason, Ridolf '97], [Ellis, Veseli '98], [Kulesza, Stirling '00],
  [Monni, Re, Torrielli '16], [Bizon, Monni, Re, Rottoli, Torrielli '17] ($\rightarrow$ see Luca's talk)

Goal: Resummation in momentum space as a complementary approach.

Factorization of transverse momentum distributions.

Factorization theorem.

$$\sigma(\vec{q}_T) = \sigma_0 H(Q,\mu) \int d^2\vec{k}_1\, d^2\vec{k}_2\, d^2\vec{k}_s\, \delta(\vec{q}_T - \vec{k}_1 - \vec{k}_2 - \vec{k}_s)$$
$$\times B(Qe^Y, \vec{k}_1, \mu, \nu) B(Qe^{-Y}, \vec{k}_2, \mu, \nu) S(\vec{k}_s, \mu, \nu)$$



[Collins,Soper,Sterman '85]
[Becher, Neubert '10]
[Echevarria, Idilbi, Scimemi '11]
[Chiu, Jain, Neill, Rothstein '12]

- Hard function: Describes hard process, e.g. $gg \to H \to VV$
- Beam functions: Describe collinear radiation along beam axes
  Often referred to as transverse-momentum dependent PDFs (TMDPDFs)
- Soft function: Describes isotropic, soft radiation
- Corrections are power suppressed by q_T/Q

Resummation of large logarithms.

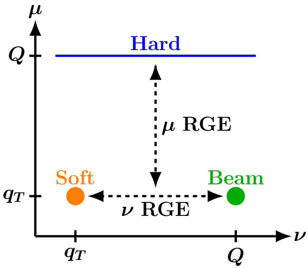
- Hard, beam and soft functions contain UV and rapidity divergences
- Renormalization induces unphysical scales μ and ν (ζ in CSS)

$$\sigma(\vec{q}_T) \sim \sigma_0 H(Q,\mu)[B(\mu,\nu) \otimes B(\mu,\nu) \otimes S(\mu,\nu)](\vec{q}_T)$$

- Large Sudakov logarithms split into

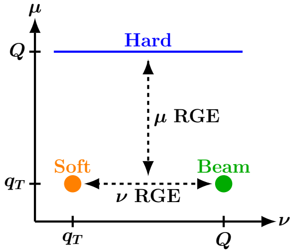
$$\ln^2 \frac{Q}{q_T} = \ln^2 \frac{Q}{\mu} + 2 \ln \frac{q_T}{\mu} \ln \frac{\nu}{Q} + \ln \frac{q_T}{\mu} \ln \frac{\mu\, q_T}{\nu^2}$$

- All-order logarithmic structure encoded in renormalization group equations (RGE)
- RGEs allow to resum large logarithms individually in hard, beam and soft functions
  - Logarithms in spectrum can be resummed to all orders in α_s
- Resummation accuracy fully specified by boundary terms / anomalous dimensions



RG structure of the cross section.

RGEs capture all-order logarithmic structure:



- μ-RGE:
$$\mu \frac{\mathrm{d}H(Q,\mu)}{\mathrm{d}\mu} = \gamma_H \, H(Q,\mu)$$
$$\mu \frac{\mathrm{d}B(\omega,\vec{q}_T,\mu,\nu)}{\mathrm{d}\mu} = \gamma_B \, B(\omega,\vec{q}_T,\mu,\nu)$$
$$\mu \frac{\mathrm{d}S(\vec{q}_T,\mu,\nu)}{\mathrm{d}\mu} = \gamma_S \, S(\vec{q}_T,\mu,\nu)$$
- ν-RGE (RRGE):
$$\nu \frac{\mathrm{d}B(\omega,\vec{q}_T,\mu,\nu)}{\mathrm{d}\nu} = -\frac{1}{2}\int \mathrm{d}^2\vec{k}_T \, \gamma_\nu(\vec{q}_T-\vec{k}_T,\mu) B(\omega,\vec{k}_T,\mu,\nu)$$
$$\nu \frac{\mathrm{d}S(\vec{q}_T,\mu,\nu)}{\mathrm{d}\nu} = \int \mathrm{d}^2\vec{k}_T \, \gamma_\nu(\vec{q}_T-\vec{k}_T,\mu) S(\vec{k}_T,\mu,\nu)$$
$$\equiv \gamma_\nu(\vec{q}_T,\mu) \otimes S(\vec{q}_T,\mu,\nu)$$
- Commutativity of μ and ν RGE:
$$\mu \frac{\mathrm{d}\gamma_\nu(\vec{q}_T,\mu)}{\mathrm{d}\mu} = \nu \frac{\mathrm{d}\gamma_S}{\mathrm{d}\nu}\delta(\vec{q}_T) = -4\Gamma_C[\alpha_s(\mu)]\delta(\vec{q}_T)$$

RG structure of the cross section.

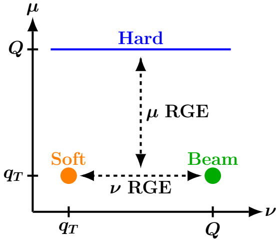
- All-order logarithmic structure encoded in the RGEs:

$$\mu \frac{\mathrm{d} f(\vec{q}_T, \mu\, \nu)}{\mathrm{d}\mu} = \gamma_{\mu,f}\, f(\vec{q}_T, \mu, \nu)$$
$$\nu \frac{\mathrm{d} f(\vec{q}_T, \mu\, \nu)}{\mathrm{d}\nu} = \gamma_{\nu,f}(\vec{q}_T, \mu) \otimes f(\vec{q}_T, \mu\, \nu)$$
$$\mu \frac{\mathrm{d}\gamma_\nu(\vec{q}_T, \mu)}{\mathrm{d}\mu} = -4\Gamma_C \delta(\vec{q}_T)$$

- *Equivalent* set of RGEs in all RG-based formalisms (CSS, SCET, . . .)
  - Differ only in rapidity regularization ($\zeta, \nu, \dots$)
  - Some approaches combine beam and soft function into TMDPDF
- Factorization proven to all orders
  $\rightarrow$ not restricted to particular order $\mathbf{N^nLO + N^nLL}$
- RGEs encode the exponential structure of the large logarithm $\ln(Q/q_T)$
- Accuracy of resummation fully specified by fixed-order accuracy of anomalous dimensions / boundary terms
  Ex.: NNLL = one-loop FO + two-loop Γ_C + one-loop non-cusp anom dim

Difficulties with resummation in momentum space.

Solving RGEs in Fourier space.



- Convolutions become products in Fourier space
- RGEs are easily solved:
  (Simplification: $\mu_T = \mu_B = \mu_S$)

$$\sigma(\vec{q}_T) = \sigma_0 \int \mathrm{d}^2\vec{b}_T\, e^{i\vec{b}_T\cdot\vec{q}_T}$$

Fixed order boundary $\longrightarrow$ $\times\, H(Q,\mu_H)\tilde{B}(\omega,\vec{b}_T,\mu_T,\nu_B)^2\,\tilde{S}(\vec{b}_T,\mu_T,\nu_S)$

RG evolution $\longrightarrow$ $\times\, \underbrace{\exp\left[\int_{\mu_H}^{\mu_T}\frac{\mathrm{d}\mu'}{\mu'}\gamma_H(Q,\mu')\right]}_{\mu\text{ evolution}}\, \underbrace{\exp\left[\ln\frac{\nu_S}{\nu_B}\tilde{\gamma}_\nu(\vec{b}_T,\mu_T)\right]}_{\nu\text{ evolution}}$

- Resummation in Fourier space

$$\mu_H = Q\,,\quad \mu_T = 1/b_T\,,\quad \nu_B = Q\,,\quad \nu_S = 1/b_T$$

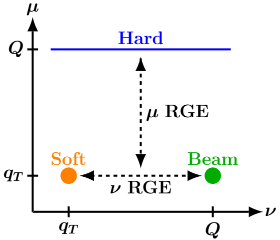
- “Natural” scales in momentum space (?)

$$\mu_H = Q\,,\quad \mu_T = q_T\,,\quad \nu_B = Q\,,\quad \nu_S = q_T$$

First attempt at momentum space resummation.

- "Natural" choice in momentum space:

$$\mu_H = Q\,, \quad \mu_T = q_T$$
$$\nu_B = Q\,, \quad \nu_S = q_T$$



- At LL, this gives

$$\sigma(\vec{q}_T) = \frac{\sigma_0}{q_T^2} \exp\Big[-\int_{q_T}^{Q} \frac{\mathrm{d}\mu'}{\mu'} \gamma_H(Q, \mu')\Big] e^{-2\gamma_E \omega} \frac{\Gamma(1-\omega)}{\Gamma(\omega)}$$
$$\omega = 2\Gamma_C[\alpha_s(q_T)] \ln \frac{Q}{q_T}$$

- The spectrum contains a divergence when $\Gamma_C[\alpha_s(q_T)] = \ln^{-1} \dfrac{Q^2}{q_T^2}$
  - For $Q = m_H$: $q_T \approx 8\ \mathrm{GeV}\ (\Gamma_C \sim C_A)$
  - For $Q = m_Z$: $q_T \approx 2\ \mathrm{GeV}\ (\Gamma_C \sim C_F)$

- Simple momentum space resummation ill-defined!
  Already noted in [Frixione, Nason, Ridolfi '97; Chiu, Jain, Neill, Rothstein '12]

Problems in the momentum space resummation.

- Divergence arises from ν evolution of soft function:

$$S(\vec{p}_T, \mu, \nu_B) = \int d^2\vec{b}_T \, e^{i\vec{b}_T \cdot \vec{p}_T} \, \tilde{S}(\vec{b}_T, \mu, \nu_S) \exp\left[\ln \frac{\nu_B}{\nu_S} \tilde{\gamma}_\nu(\vec{b}_T, \mu)\right]$$

$$= S(\vec{p}_T, \mu, \nu_S) + \sum_{n=1}^{\infty} \frac{1}{n!} \ln^n \frac{\nu_B}{\nu_S} [(\gamma_\nu \otimes^n) \otimes S](\vec{p}_T, \mu, \nu_S)$$

- Obviously fulfills RGE

$$\nu \frac{d}{d\nu} S(\vec{p}_T, \mu, \nu) = \int d^2\vec{k}_T \, \gamma_\nu(\vec{p}_T - \vec{k}_T, \mu) S(\vec{k}_T, \mu, \nu)$$

- A priori, this only *shifts* $\ln(p_T/\nu_S)$ into $\ln(p_T/\nu_B)$
- Resummation assumes that $S(\vec{p}_T, \mu, \nu_S)$ has no large logs, i.e.

$$S(\vec{p}_T, \mu, \nu_S) = \delta(\vec{p}_T) + \cdots$$

$\Rightarrow$ All logs are predicted through RG evolution

- Does $\nu_S = q_T$ eliminate all logarithms in the boundary $S(\vec{p}_T, \mu, \nu_S)$?

Problems in the momentum space resummation.

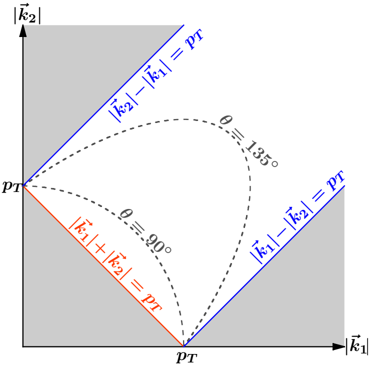
Illustration in momentum space:

- Investigate the first convolution in more detail:

$$\ln \frac{\nu_B}{\nu_S} \int d^2\vec{k}_1 \, d^2\vec{k}_2 \, \gamma_\nu(\vec{k}_1, \mu) \, S(\vec{k}_2, \mu, \nu_S) \, \delta(\vec{p}_T - \vec{k}_1 - \vec{k}_2)$$

- Which momenta $|\vec{k}_1|, |\vec{k}_2|$ contribute to the convolution?

- Kinematically forbidden



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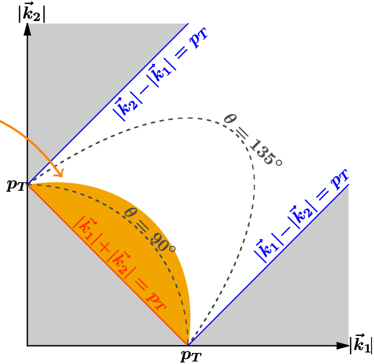
- Soft contributions $k_1 \sim k_2 \sim p_T$
  - Correctly described by $\nu_S \sim p_T$
  - Induces large $\ln(\nu_B/\nu_S)$



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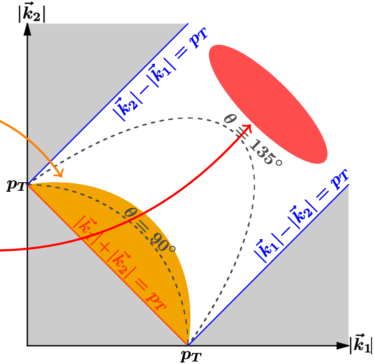
  - Correctly described by $\nu_S \sim p_T$
  - Induces large $\ln(\nu_B/\nu_S)$
- Hard contributions $k_1, k_2 \gg p_T$
  - Contributes due to kinematic cancellation $\vec{k}_1 + \vec{k}_2 = \vec{p}_T$
  - Should not give a large log...
  - but receives spurious $\ln \dfrac{\nu_B}{\nu_S}$



Problems in the momentum space resummation.

- The first convolution

$$\ln\frac{\nu_B}{\nu_S}\int d^2\vec{k}_1 d^2\vec{k}_2\,\gamma_\nu(\vec{k}_1,\mu)\,S(\vec{k}_2,\mu,\nu_S)\,\delta(\vec{p}_T-\vec{k}_1-\vec{k}_2)$$

- should actually behave as

$$\int d^2\vec{k}_1 d^2\vec{k}_2\;\underbrace{\ln\frac{\nu_B}{\nu_S\sim k_1}}_{\text{Rapidity logarithm of emission } \vec{k}_1}\;\gamma_\nu(\vec{k}_1,\mu)\,\underbrace{S(\vec{k}_2,\mu,\nu_S\sim k_2)}_{\text{Minimize logs } \ln(k_2/\nu_s) \text{ inside } S(\vec{k}_2,\mu,\nu_S)}\,\delta(\vec{p}_T-\vec{k}_1-\vec{k}_2)$$

- Can not be achieved with simple exponential

$$S(\vec{p}_T,\mu,\nu_B)=S(\vec{p}_T,\mu,\nu_S)+\sum_{n=1}^{\infty}\frac{1}{n!}\ln^n\frac{\nu_B}{\nu_S}[(\gamma_\nu\otimes^n)\otimes S](\vec{p}_T,\mu,\nu_S)$$

  - Requires a “generalized” exponential solution

- Soft function is a distribution → How to set $\nu_S=k_2$?
  - Requires “distributional scale setting”, see 1611.08610

q_T Resummation in distribution space.

Strategy.

- Focus on the soft function $S(\vec{p}_T, \mu, \nu)$
- Beam and hard functions are analogous (and much simpler)
- Soft rapidity anomalous dimension γ_ν is governed by cusp anom. dim

$$\mu \frac{\mathrm{d}\gamma_\nu(\vec{p}_T, \mu)}{\mathrm{d}\mu} = -4\Gamma_C[\alpha_s(\mu)]\delta(\vec{p}_T) \tag{1}$$

  - Crucial input to guarantee path independence of (μ, ν)-running

- Soft function evolution is governed by

$$\mu \frac{\mathrm{d}S(\vec{p}_T, \mu, \nu)}{\mathrm{d}\mu} = \gamma_S(\mu, \nu)\, S(\vec{p}_T, \mu, \nu) \tag{2}$$

$$\nu \frac{\mathrm{d}S(\vec{p}_T, \mu, \nu)}{\mathrm{d}\nu} = (\gamma_\nu \otimes S)(\vec{p}_T, \mu, \nu) \tag{3}$$

- Equations increase in complexity $\rightarrow$ solve step by step

Rapidity anomalous dimension γ_ν.

- Formal solution of rapidity anomalous dimension γ_ν:

$$\gamma_\nu(\vec{p}_T, \mu) = \gamma_\nu(\vec{p}_T, \mu_0) - \delta(\vec{p}_T) \int_{\mu_0}^{\mu} \frac{\mathrm{d}\mu'}{\mu'} 4\Gamma_C[\alpha_s(\mu')]$$

- Boundary is distributionally minimized with $\mu_0 = p_T|_+$:

$$\gamma_\nu(\vec{p}_T, \mu) = \left[\frac{4\Gamma_C[\alpha_s(p_T)]}{2\pi p_T^2}\right]_+^{\mu} + \left[\frac{1}{2\pi p_T^2} \frac{\mathrm{d}\gamma_\nu[\alpha_s(p_T)]}{\mathrm{d}\ln p_T}\right]_+^{\xi} + \delta(\vec{p}_T)\gamma_\nu[\alpha_s(\xi)]$$

- Cusp piece: Predicted by RGE
- Noncusp piece: Fixed-order boundary terms
- Virtual corrections to real emission $\vec{p}_T$ resummed in $\alpha_s(p_T)$

Soft μ evolution.

- μ evolution:

$$\mu \frac{\mathrm{d}S(\vec{p}_T, \mu, \nu)}{\mathrm{d}\mu} = \gamma_S(\mu, \nu)\, S(\vec{p}_T, \mu, \nu)$$

- Formal solution:

$$S(\vec{p}_T, \mu, \nu_0) = S(\vec{p}_T, \mu_0, \nu_0) \exp\left[\int_{\mu_0}^{\mu} \frac{\mathrm{d}\mu'}{\mu'} \gamma_S(\mu', \nu_0)\right]$$

- To minimize boundary term: $\mu_0 = \nu_0 = p_T|_+$

$$S(\vec{p}_T, \mu, p_T|_+) = \delta(\vec{p}_T) S[\alpha_s(\mu)] + \left[\frac{1}{2\pi p_T} \frac{\mathrm{d}}{\mathrm{d}p_T} S[\alpha_s(p_T)] \exp\left\{\int_{p_T}^{\mu} \frac{\mathrm{d}\mu'}{\mu'} \gamma_S(\mu', p_T)\right\}\right]_+^{\mu}$$

- Boundary term $S[\alpha_s(\mu)]$ arises as coefficient of $\delta(\vec{p}_T)$

Soft ν evolution.

- Rapidity RGE:

$$\nu \frac{\mathrm{d}S(\vec{p}_T, \mu, \nu)}{\mathrm{d}\nu} = (\gamma_\nu \otimes S)(\vec{p}_T, \mu, \nu)$$

- Correct solution is a "generalized" exponential

$$\begin{aligned} S(\vec{p}_T, \mu, \nu) = {} & S(\vec{p}_T, \mu, p_T|_+) \\ & + \int_{p_T|_+}^{\nu} \frac{\mathrm{d}\nu_1}{\nu_1} \int \mathrm{d}^2\vec{k}_1 \, \gamma_\nu(\vec{p}_T - \vec{k}_1, \mu) S(\vec{k}_1, \mu, k_1|_+) \\ & + \int_{p_T|_+}^{\nu} \frac{\mathrm{d}\nu_1}{\nu_1} \int \mathrm{d}^2\vec{k}_1 \, \gamma_\nu(\vec{p}_T - \vec{k}_1, \mu) \\ & \quad \times \int_{k_1|_+}^{\nu_1} \frac{\mathrm{d}\nu_2}{\nu_2} \int \mathrm{d}^2\vec{k}_2 \, \gamma_\nu(\vec{k}_1 - \vec{k}_2, \mu) S(\vec{k}_2, \mu, k_2|_+) \\ & + \cdots \end{aligned}$$

- Fulfills the rapidity RGE ✓
- $\nu = p_T|_+$ reproduces correct boundary $S(\vec{p}_T, \mu, p_T|_+)$

RG-evolved cross section.

$$\sigma(\vec{q}_T) = \sigma_0\, H(Q,\mu_H) \frac{1}{2\pi q_T}\frac{\mathrm{d}}{\mathrm{d}q_T}\int_{|\vec{p}_T|\le q_T} \mathrm{d}^2\vec{p}_T$$

← distributional scale setting

$$\times \exp\left[\int_{\mu_H}^{\mu_T}\frac{\mathrm{d}\mu'}{\mu'}\gamma_H(Q,\mu')\right]\int \mathrm{d}^2\vec{k}_a\,\mathrm{d}^2\vec{k}_b\,\mathrm{d}^2\vec{k}_s\,\delta(\vec{p}_T-\vec{k}_a-\vec{k}_b-\vec{k}_s)$$

$$\times \int \mathrm{d}^2\vec{k}_s' \Big[\delta(\vec{k}_s-\vec{k}_s')$$

$$+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\int_{k_{i-1}|_+}^{\nu_{i-1}}\frac{\mathrm{d}\nu_i}{\nu_i}\int \mathrm{d}^2\vec{k}_i\,\gamma_\nu(\vec{k}_{i-1}-\vec{k}_i,\mu_T)\,\delta\Big(\vec{k}_s-\vec{k}_s'-\sum_i\vec{k}_i\Big)\Big]$$

← ν evolution

$$\times B_a(\omega_a,\vec{k}_a,\mu_T,\nu_a)\,B_b(\omega_b,\vec{k}_b,\mu_T,\nu_b)\,S(\vec{k}_s',\mu_T,k_s'|_+)$$

← ν-logs minimized

- Complicated iterative distributional structure
- Intrinsically nonperturbative due to $\gamma_\nu(\vec{k}_T,\mu)\sim\alpha_s(k_T)$
  - Nonperturbative effects suppressed by $\mathcal{O}(\Lambda^2_{\rm QCD}/q_T^2)$, but require nonperturbative model
- No numerical evaluation available yet

Verification: LL cross section without α_s-running

Solution in distribution space:

$$\begin{aligned}\sigma(\vec{q}_T) = \sigma_0 \frac{1}{2\pi q_T}\frac{\mathrm{d}}{\mathrm{d}q_T}\theta(q_T) f_a(\omega_a, q_T)\, f_b(\omega_b, q_T) \exp\left[-\frac{\Gamma_C}{2}\ln^2\frac{Q^2}{q_T^2}\right] \\ \times \Bigg[1 - 2\Gamma_C^2\zeta_3 \ln\frac{Q^2}{q_T^2} + \Gamma_C^3\left(\frac{2\zeta_3}{3}\ln^3\frac{Q^2}{q_T^2} + 6\zeta_5\ln\frac{Q^2}{q_T^2}\right) \\ + \Gamma_C^4\left(-4\zeta_5\ln^3\frac{Q^2}{q_T^2} + 10\zeta_3^2\ln^2\frac{Q^2}{q_T^2} - 30\zeta_7\ln\frac{Q^2}{q_T^2}\right) \\ + \mathcal{O}(\Gamma_C^5)\Bigg]\end{aligned}$$

- Exponential resums $\ln(Q/q_T)$ at LL
- Many apparent-subleading terms arise from rapidity evolution
- These have no simple exponential structure

Verification: LL cross section without α_s-running.

Solution in Fourier space:

$$\sigma(\vec{q}_T) = \sigma_0 \frac{1}{2\pi q_T} \frac{\mathrm{d}}{\mathrm{d}q_T} \theta(q_T) f_a(\omega_a, q_T)\, f_b(\omega_b, q_T) \exp\left[-\frac{\Gamma_C}{2} \ln^2 \frac{Q^2}{q_T^2}\right]$$

$$\times \left[1 - 2\Gamma_C^2 \zeta_3 \ln \frac{Q^2}{q_T^2} + \Gamma_C^3 \left(\frac{2\zeta_3}{3} \ln^3 \frac{Q^2}{q_T^2} + 6\zeta_5 \ln \frac{Q^2}{q_T^2} - \frac{10}{3}\zeta_3^2\right)\right.$$

$$+ \Gamma_C^4 \left(-4\zeta_5 \ln^3 \frac{Q^2}{q_T^2} + 10\zeta_3^2 \ln^2 \frac{Q^2}{q_T^2} - 30\zeta_7 \ln \frac{Q^2}{q_T^2} + 28\zeta_3\zeta_5\right)$$

$$\left. + \mathcal{O}(\Gamma_C^5)\right]$$

$$= \sigma_0\, f_a(\omega_a, \mu)\, f_b(\omega_b, \mu) \int \frac{\mathrm{d}^2\vec{b}_T}{(2\pi)^2} e^{i\vec{b}_T\cdot\vec{q}_T} \exp\left[-\frac{\Gamma_C}{2} \ln^2 \frac{Q^2 b_T^2}{4e^{-2\gamma_E}}\right]$$

- Simple Sudakov exponential in Fourier space $\to$ solution well-defined
- Induces same apparent-subleading terms as distributional solution
- Differ only by *constant* terms
  $\to$ Intrinsically different boundary condition than distribution space

Verification: LL cross section without α_s-running.

Naive solution in momentum space:

$$\sigma(\vec{q}_T) = \sigma_0 \frac{1}{2\pi q_T} \frac{\mathrm{d}}{\mathrm{d}q_T} \theta(q_T) f_a(\omega_a, q_T)\, f_b(\omega_b, q_T) \exp\left[-\frac{\Gamma_C}{2} \ln^2 \frac{Q^2}{q_T^2}\right]$$
$$\times \left[1 + \frac{2}{3}\Gamma_C^3 \ln^3 \frac{Q^2}{q_T^2} \zeta_3 + \mathcal{O}(\Gamma_C^5)\right]$$

- Naive solution misses many apparently subleading terms
- These are crucial to cancel the observed divergence:

  Hence, we find the peculiar feature that apparent-NNLL and higher terms cancel the divergence caused by the apparent-NLL terms in the strict LL spectrum.

- Previous attempts tried to obtain an exponential form by neglecting apparently subleading terms ✗ [Frixione, Nason, Ridolfi '97] [Ellis, Veseli '98]
- Resummation accuracy can not be specified by counting $\ln(Q/q_T)$!

Comparison to RadiSH.

- [Monni, Re, Torrielli ’16], [Bizon, Monni, Re, Rottoli, Torrielli ’17] recently carried out resummation in momentum space using the ARES framework:

$$\sigma(\vec{q}_T) = \sigma_0 \int \mathrm{d}^2\vec{k}_1 \, \frac{R'(k_1)}{2\pi k_1^2} e^{-R(\epsilon k_1)} \longleftarrow \vec{k}_1 \text{ hardest emission}$$

$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_{\epsilon k_1 < |\vec{k}_i| < k_1} \mathrm{d}^2\vec{k}_i \, \frac{R'(k_i)}{2\pi k_i^2} \, \delta\Big(\vec{q}_T - \sum\nolimits_j \vec{k}_j\Big)$$

- Comparison to our result:
  - μ evolution: $R(\epsilon k_T) = \int_{\epsilon k_T}^{Q} \frac{\mathrm{d}\mu'}{\mu'} \gamma_H(Q, \mu')$
  - ν evolution: $\frac{R'(k_i)}{2\pi k_i^2} = \int_{k_i}^{Q} \frac{\mathrm{d}\nu'}{\nu'} \gamma_\nu(\vec{k}_i, \mu)$
  - ϵ acts as IR-regulator (similar to plus distributions)
- Both approaches are closely related

Comparison to RadiSH (NLL).

- [Monni, Re, Torrielli '16], [Bizon, Monni, Re, Rottoli, Torrielli '17] recently carried out resummation in momentum space using the ARES framework:

$$\sigma(\vec{q}_T) = \sigma_0 \int \mathrm{d}^2\vec{k}_1 \, \frac{R'(k_1)}{2\pi k_1^2} e^{-R(k_1)+\ln(\epsilon)R'(k_1)} \times \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_{\epsilon k_1 < |\vec{k}_i| < k_1} \mathrm{d}^2\vec{k}_i \, \frac{R'(k_1)}{2\pi k_i^2} \, \delta\Big(\vec{q}_T - \sum\nolimits_j \vec{k}_j\Big)$$

- In practice: expand all $\vec{k}_i$ around hardest emission $\vec{k}_1$
- Resums $\ln(Q/k_1)$ rather than $\ln(Q/q_T)$
  - Formally equivalent?
- Great simplification: Avoids most nonperturbative effects
- Formally subleading difference to our method
  - It will be interesting to check differences numerically

Conclusion.

Conclusion.

Resummation in momentum space:

- All-order logarithmic structure encoded in RGEs
- Proper treatment of kinematic cancellations requires *distributional* solution of RGEs
- Solved in principle: Allows resummation to arbitrary order $\mathbf{N^n LL}$
  - Accuracy fully specified in terms of anomalous dimension $\leftrightarrow$ no classification in $\ln(Q/q_T)$ possible
- Complicated numerical implementation $\to$ work in progress

Phenomenological impact:

- Boundary terms intrinsically different from Fourier space resummation $\Rightarrow$ expect insight into (non)perturbative uncertainties
- $\vec{q}_T$ spectrum is intrinsically nonperturbative (no flaw of $\vec{b}_T$ space), but nonperturbative contributions are suppressed as $\mathcal{O}(\Lambda^2_{\mathrm{QCD}}/q_T^2)$
- Numerical study will be interesting (see also RadiSH)

Conclusion.

Resummation in momentum space:

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Phenomenological impact:

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Thank you for your attention!

Backup slides.

Problems in the momentum space resummation.

Origin of divergence in Fourier space:

$$S(\vec{p}_T, \mu, \nu_B) = \int d^2\vec{b}_T \, e^{i\vec{b}_T\cdot\vec{p}_T} \, \tilde{S}(\vec{b}_T, \mu, \nu_S) \exp\left[\ln\frac{\nu_B}{\nu_S}\tilde{\gamma}_\nu(\vec{b}_T, \mu)\right]$$

- $\nu_S = p_T$ assumes $\ln(b_T\nu_S) = \ln(b_T p_T) \sim 0$ in Fourier transformation:

$$\begin{aligned}\tilde{S}(\vec{b}_T, \mu, \nu_S = p_T) &\sim 1 + \alpha_s \ln(b_T\nu_S)\ln(b_T\mu) + \cdots \\ &= 1 + \alpha_s \ln(b_T p_T)\ln(b_T\mu) + \cdots \\ &\approx 1 + \cdots\end{aligned}$$

- In fact: region $b_T \ll p_T^{-1}$ induces the explicit divergence:

$$\begin{aligned}S(\vec{p}_T, \mu, \nu_B) &\approx \int d^2\vec{b}_T \, \exp\left[-2\Gamma_C \ln\frac{\nu_B}{\nu_S}\ln(b_T^2\mu^2)\right] \\ &\sim \frac{1}{1-\omega}\,, \qquad \omega = 2\Gamma_C[\alpha_s(\mu)]\ln\frac{\nu_B}{\nu_S}\end{aligned}$$

- Divergence results from energetic emissions $b_T^{-1} \gg p_T$!

Comparison: Resummation in Fourier space.

- Toy function is governed by $\mu \frac{dF(k,\mu)}{d\mu} = -\alpha_s F(k,\mu)$
- Formal solution in Fourier space: (neglecting α_s running)

$$\tilde{F}(y,\mu) = \tilde{F}(y,\mu_0) \exp\left(\alpha_s \ln \frac{\mu_0}{\mu}\right)$$

- $\tilde{F}(y,\mu_0)$ depends on $\ln(iy\mu_0 e^{\gamma_E}) \Rightarrow$ Choose $\mu_0 = -ie^{-\gamma_E}/y$
- Transform back to distribution space:

$$F(k,\mu) = \underbrace{\frac{e^{-\gamma_E \alpha_s}}{\Gamma(1+\alpha_s)}}_{\text{Subleading correction}} \underbrace{\left\{\delta(k) + \alpha_s \left[\frac{\theta(k)}{k} \exp\left(\alpha_s \ln \frac{k}{\mu}\right)\right]_+^\mu\right\}}_{\text{distributional solution}}$$

- Fourier spaces induces different boundary term:

$$\frac{e^{-\gamma_E \alpha_s}}{\Gamma(1+\alpha_s)} = 1 - \frac{\pi^2}{12}\alpha_s^2 + \cdots$$

Resummation in distribution space / Fourier space intrinsically probe different boundary conditions!

Comparison to DDT-formula.

- Structure of the LL resummed cross section by [Dokshitzer, Diakonov, Troian '80]

$$\sigma^{\mathrm{DDT}}(\vec{q}_T) = \sigma_0 \frac{\mathrm{d}}{\mathrm{d}q_T^2} f_a(\omega_a, q_T)\, f_b(\omega_b, q_T)\, e^{\mathcal{S}(Q,q_T)}$$

- Comparison to LL in strict RGE counting (not counting $\ln(Q/q_T)$):

$$\sigma^{\mathrm{LL}}(\vec{q}_T) = \sigma_0 \frac{1}{2\pi q_T}\frac{\mathrm{d}}{\mathrm{d}q_T} f_a(\omega_a, \mu_T)\, f_b(\omega_b, \mu_T) \int\limits_{|\vec{p}_T|\leq q_T} \mathrm{d}^2\vec{p}_T$$

$$\times \exp\left[\int_{\mu_H}^{\mu_T} \frac{\mathrm{d}\mu'}{\mu'}\gamma_H(Q,\mu')\right] \left[\int \mathrm{d}^2\vec{k}_s \delta(\vec{k}_s - \vec{k}_s')\right.$$

$$\left. + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\int_{k_{i-1}|_+}^{\nu_{i-1}} \frac{\mathrm{d}\nu_i}{\nu_i}\int \mathrm{d}^2\vec{k}_i\, \gamma_\nu(\vec{k}_{i-1} - \vec{k}_i, \mu_T)\, \delta\Big(\vec{k}_s - \vec{p}_T - \sum_i \vec{k}_i\Big)\right]$$

$$\times \left(\delta(\vec{k}_s) + \left[\frac{1}{2\pi k_s}\frac{\mathrm{d}}{\mathrm{d}k_s} \exp\left\{\int_{k_s}^{\mu_T}\frac{\mathrm{d}\mu'}{\mu'} 4\Gamma_{\mathrm{C}}[\alpha_s(\mu')]\ln\frac{\mu'}{k_s}\right\}\right]_+^{\mu_T}\right)$$

- Simple structure of DDT-formula can not hold due to rapidity logarithms ✗

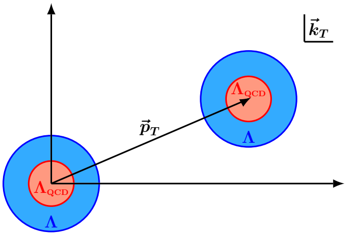
Illustration of soft ν evolution.

- Solve ν-RGE
$$\nu \frac{\mathrm{d}S(\vec{p}_T, \mu, \nu)}{\mathrm{d}\nu} = (\gamma_\nu \otimes S)(\vec{p}_T, \mu, \nu)$$
starting with boundary term $S(\vec{p}_T, \mu, \nu = p_T|_+)$
- To illustrate solution, expand
$$S(\vec{p}_T, \mu, \nu) = \sum_{n=0}^{\infty} S^{[n]}(\vec{p}_T, \mu, \nu)$$
  - $S^{[n]}$ = soft function after n real emissions
  - γ_ν counts as one real emission
- ν-RGE now becomes
$$\nu \frac{\mathrm{d}}{\mathrm{d}\nu} S^{[n]}(\vec{p}_T, \mu, \nu) = (\gamma_\nu \otimes S^{[n-1]})(\vec{p}_T, \mu, \nu)$$
- Recursive solution:
$$S^{[n]}(\vec{p}_T, \mu, \nu) = S^{[n]}(\vec{p}_T, \mu, \nu_0) + \int_{\nu_0}^{\nu} \frac{\mathrm{d}\nu'}{\mu'} (\gamma_\nu \otimes S^{[n-1]})(\vec{p}_T, \mu, \nu')$$
- This requires $\nu_0 = p_T|_+$ at each order n!

Perturbativity of ν-kernel.

- The ν-kernel involves convolutions $\gamma_\nu \otimes^n$
- Problematic due to Landau pole in: $\gamma_\nu(\vec{k}_T, \mu) \sim \left[\frac{4\Gamma_C[\alpha_s(k_T)]}{2\pi k_T^2}\right]_+^\mu$
  - Expect $(\gamma_\nu \otimes \gamma_\nu)(\vec{p}_T, \mu) = \int \mathrm{d}^2\vec{k}_T \, \gamma_\nu(\vec{p}_T - \vec{k}_T, \mu)\gamma_\nu(\vec{k}_T, \mu)$ to be non-perturbative
- $p_T, \mu \gg \Lambda_{\text{QCD}}$: Landau pole effectively regulated by plus prescription
- Illustration:

$$\int \mathrm{d}^2\vec{k}_T \, \gamma_\nu(\vec{p}_T - \vec{k}_T, \mu)\gamma_\nu(\vec{k}_T, \mu)$$
$$\sim -2\left[\frac{\Gamma_C[\vec{p}_T]}{p_T^2} + \mathcal{O}\left(\frac{\Lambda}{p_T}\right)\right]_+^\mu \int_\Lambda^\mu \frac{\mathrm{d}k_T}{k_T}\Gamma_C[\alpha_s(k_T)]$$
$$+ (\gamma_\nu \otimes \gamma_\nu)\Big|_{\mathbb{R}^2\setminus(B_\Lambda(\vec{0})\cup B_\Lambda(\vec{p}_T))}(\vec{p}_T, \mu)$$



Implementation of profiles.

- In practice: Vary scales to test perturbative uncertainties

- Also want to transition between
  - Canonical scale setting: $\mu_0 = p_T|_+ \,, \quad p_T \ll Q$
  - Fixed order regime: $\mu_0 = \mu_{\rm FO} \,, \quad p_T \sim Q$

- Implementation through a profile $\mu_0(p_T)$ with distributional scale setting

- Example: Rapidity anomalous dimension

$$\gamma_\nu(\vec p_T, \mu) = \left\{ \gamma_\nu^{\rm FO}(\vec p_T, \mu_0) - \delta(\vec p_T) \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} 4\Gamma_{\rm C}[\alpha_s(\mu')] \right\}_{\mu_0 = \mu_0(p_T)|_+}$$

Profile function:

  - Canonical scale setting: $\mu_0(p_T) \propto p_T \,, \quad p_T \ll Q$
  - Fixed order regime: $\mu_0(p_T) \propto \mu_{\rm FO} \,, \quad p_T \sim Q$

Example: Implementation of profiles for γ_ν.

- Resum γ_ν at lowest order (i.e. keep only β_0, Γ_0)
- together with the full $\mathcal{O}(\alpha_s)$ boundary term $2\Gamma_0 \mathcal{L}_0(\vec{p}_T, \mu) \frac{\alpha_s(\mu)}{4\pi}$
- Result:

$$\gamma_\nu^{(\mathrm{N})\mathrm{LL}(0)}(\vec{p}_T, \mu) = 2\Gamma_0\, \mathcal{L}_0(\vec{p}_T, \mu)$$

$$\gamma_\nu^{(\mathrm{N})\mathrm{LL}(1)}(\vec{p}_T, \mu) = -2\beta_0\Gamma_0 \frac{1}{\pi\mu^2}\left[\frac{\mu^2}{\vec{p}_T^{\,2}} \ln\frac{\mu_0(p_T)^2}{\mu^2}\right]_+^\mu + 2\beta_0\Gamma_0 \frac{1}{\pi\mu^2}\left[\frac{\mu^2}{\vec{p}_T^{\,2}} \ln\frac{\mu_0(p_T)^2}{p_T^2} \frac{\mathrm{d}\ln\mu_0(p_T)}{\mathrm{d}\ln p_T}\right]_+^\mu + 4\Gamma_0\beta_0 \ln^2\frac{\mu}{\mu_0(\mu)}\delta(\vec{p}_T)$$

- Full result at $\mathcal{O}(\alpha_s^2)$:

$$\gamma_\nu^{(1)}(\vec{p}_T, \mu) = -2\beta_0\Gamma_0 \frac{1}{\pi\mu^2}\left[\frac{\mu^2}{\vec{p}_T^{\,2}} \ln\frac{p_T^2}{\mu^2}\right]_+^\mu + 2\Gamma_1 \frac{1}{\pi\mu^2}\left[\frac{\mu^2}{\vec{p}_T^{\,2}}\right]_+^\mu + \gamma_{\nu\,1}\delta(\vec{p}_T)$$

  - Vanishes for canonical scale $\mu_0(x) = x \to$ probes subleading terms
  - Reproduces exact result to this order for canonical scale $\mu_0(x) = x$

Nonperturbative modelling of γ_ν.

- Consider for simplicity only $\gamma_\nu(\vec{p}_T, \mu) = \left[\frac{4\Gamma_C[\alpha_s(p_T)]}{2\pi p_T^2}\right]_+^\mu + \cdots$
- Split using a profile function $\mu_0(p_T)$:

$$\gamma_\nu(\vec{p}_T, \mu) = \left[\frac{1}{2\pi p_T^2} 4\Gamma_C[\alpha_s(\mu_0(p_T))]\right]_+^\mu + \left[\frac{1}{2\pi p_T^2}\Big(4\Gamma_C[\alpha_s(p_T)] - 4\Gamma_C[\alpha_s(\mu_0(p_T))]\Big)\right]_+^\mu$$

- For suitable profiles μ_0, we can expand in moments:

$$\gamma_\nu(\vec{p}_T, \mu) = \left[\frac{1}{2\pi p_T^2} 4\Gamma_C[\alpha_s(\mu_0(p_T))]\right]_+^\mu + \sum_{n=1}^\infty \Omega_n \Delta^n \delta(\vec{p}_T)$$

- Moments become more intuitive in $\vec{b}_T$-space:

$$\int \mathrm{d}^2\vec{p}_T\, e^{i\vec{p}_T\cdot\vec{b}_T} \sum_{n=1}^\infty \Omega_n \Delta^n \delta(\vec{p}_T) = \sum_{n=1}^\infty \Omega_n (-b_T^2)^n$$

- Leading nonperturbative effect on spectrum is a Gaussian:

$$\sigma(\vec{q}_T) \sim \int \mathrm{d}^2\vec{b}_T\, e^{i\vec{q}_T\cdot\vec{b}_T} H(Q) B(b_T)^2 S(b_T) e^{\ln\frac{\nu_S}{\nu_B}\tilde{\gamma}_\nu^{(\mathrm{pert})}(\vec{b}_T,\mu)} e^{-\ln\frac{\nu_S}{\nu_B}\Omega_1 b_T^2 + \ldots}$$

RG-evolved cross section (1).

distributional scale setting

$$\sigma(\vec{q}_T) = \sigma_0\, H(Q,\mu_H) \frac{1}{2\pi q_T} \frac{\mathrm{d}}{\mathrm{d}q_T} \int_{|\vec{p}_T|\le q_T} \mathrm{d}^2\vec{p}_T$$

$$\times \exp\left[\int_{\mu_H}^{\mu_T} \frac{\mathrm{d}\mu'}{\mu'}\, \gamma_H(Q,\mu')\right] \int \mathrm{d}^2\vec{k}_a\, \mathrm{d}^2\vec{k}_b\, \mathrm{d}^2\vec{k}_s\, \delta(\vec{p}_T - \vec{k}_a - \vec{k}_b - \vec{k}_s)$$

$$\times \int \mathrm{d}^2\vec{k}'_s \Bigg[\delta(\vec{k}_s - \vec{k}'_s)$$

$$+ \sum_{n=1}^{\infty} \prod_{i=1}^{n} \int_{k_{i-1}|_+}^{\nu_{i-1}} \frac{\mathrm{d}\nu_i}{\nu_i} \int \mathrm{d}^2\vec{k}_i\, \gamma_\nu(\vec{k}_{i-1} - \vec{k}_i, \mu_T)\, \delta\Big(\vec{k}_s - \vec{k}'_s - \sum_i \vec{k}_i\Big)\Bigg]$$

$$\times B_a(\omega_a, \vec{k}_a, \mu_T, \nu_a)\, B_b(\omega_b, \vec{k}_b, \mu_T, \nu_b)\, S(\vec{k}'_s, \mu_T, k'_s|_+)$$

ν evolution

ν-logs minimized

RG-evolved cross section (2).

$$B_a(\omega_a, \vec{k}_a, \mu_T, \nu_a)\, B_b(\omega_b, \vec{k}_b, \mu_T, \nu_b)\, S(\vec{k}'_s, \mu_T, k'_s|_+)$$

$$\supset \Big[\frac{1}{2\pi k_a} \frac{\mathrm{d}}{\mathrm{d}k_a} \frac{1}{2\pi k_b} \frac{\mathrm{d}}{\mathrm{d}k_b} \frac{1}{2\pi k_s} \frac{\mathrm{d}}{\mathrm{d}k_s} B(\omega_a, k_a)\, B(\omega_b, k_b)\, S[\alpha_s(k_s)]$$

$$\times \exp\Big\{ \int_{k_a}^{\mu_T} \frac{\mathrm{d}\mu'}{\mu'} \gamma_B(\omega_a, \mu', \omega_a) + \int_{k_b}^{\mu_T} \frac{\mathrm{d}\mu'}{\mu'} \gamma_B(\omega_b, \mu', \omega_b)$$

$$+ \int_{k_s}^{\mu_T} \frac{\mathrm{d}\mu'}{\mu'} \gamma_S(\mu', k_s) \Big\} \Big]_+^{\mu_T}$$

All logs minimized
(pure boundary)

Comparison to Fourier resummation.

- Fourier resummation has analogous form of [Collins, Soper, Sterman '85]:

$$\sigma(\vec{q}_T) = \sigma_0 \int \frac{d^2\vec{b}_T}{(2\pi)^2}\, e^{i\vec{b}_T\cdot\vec{q}_T} \times \exp\Big[-\int_{(1/b_T)^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\Big(\ln\frac{Q^2}{\bar{\mu}^2} A[\alpha_s(\bar{\mu})] + 2B[\alpha_s(\bar{\mu})]\Big)\Big] \times \underbrace{H(Q,Q)\tilde{B}^2(\vec{b}_T, 1/b_T, Q)\tilde{S}(\vec{b}_T, 1/b_T, 1/b_T)}_{\text{All scales minimized in Fourier space}}$$

- Relation to our notation:
  - $A(\alpha_s) = \Gamma_C(\alpha_s) + \beta(\alpha_s)\dfrac{d\tilde{\gamma}_\nu(\alpha_s)}{\alpha_s}$

    $\to$ Rapidity evolution contributes to cusp-term [Becher,Neubert '10]
  - $B(\alpha_s) = \gamma_H(\alpha_s) - \tilde{\gamma}_\nu(\alpha_s)$
- Remark: Beam and soft functions are often combined into TMDPDFs
- In practice: $\ln(Q^2 b^2) \to \ln(1 + Q^2 b^2)$ differs from canonical resummation, but suppresses small $\vec{b}_T \to 0$, e.g. in DYRes [Catani, de Florian, Ferrera, Grazzini '15 ...], HRes [de Florian, Ferrera, Grazzini, Tommasini '12 ...]

Comparison to partial Fourier resummation.

- Many groups employ hybrid scale choice μ_T [Becher, Neubert, Wilhelm '12 '13], [D'Alesio, Echevarria, Melis, Scimemi '14], [Echevarria, Kasemets, Mulders, Pisano '15]

$$\mu_H = Q\,, \mu_B = \mu_T\,, \; \mu_S = \mu_T$$
$$\nu_B = Q\,, \; \nu_S \sim 1/b_T$$

- where $\mu_T \sim Q_0 + q_T$ is chosen in momentum space and $Q_0 \approx 2\ \mathrm{GeV}$ [D'Alesio, Echevarria, Melis, Scimemi '14] / $Q_0 \approx 8\ \mathrm{GeV}$ [Becher, Neubert, Wilhelm '12]
- RG-evolved cross section:

$$\sigma(\vec{q}_T) = \sigma_0 \int \frac{\mathrm{d}^2\vec{b}_T}{(2\pi)^2}\, e^{i\vec{b}_T\cdot\vec{q}_T}$$
$$\times \exp\Big[-\int_{\mu_T}^{Q}\frac{\mathrm{d}\mu'}{\mu'}\gamma_H(Q,\mu')\Big]\exp\Big[-\ln(b_T Q)\tilde{\gamma}_\nu(\vec{b}_T,\mu_T)\Big]$$
$$\times H(Q,Q)\tilde{B}^2(\vec{b}_T,\mu_T,Q)\tilde{S}(\vec{b}_T,\mu_T,1/b_T)$$

  - ν-scale chosen in Fourier space $\rightarrow$ no spurious divergence
  - μ-scale chosen in physical space $\rightarrow$ no nonperturbative effects
- Differs from canonical resummation for small $\vec{q}_T$

Distributional scale setting.

(For simplicity: for one-dimensional distributions)

Toy example.

- Toy function $F(k, \mu)$ containing logarithms $\ln(k/\mu)$
- Logarithmic structure governed by toy RGE:

$$\mu \frac{\mathrm{d}F(k,\mu)}{\mathrm{d}\mu} = -\alpha_s F(k,\mu)$$

- Formal solution: (neglecting α_s running)

$$F(k,\mu) = F(k,\mu_0) \exp\left(\alpha_s \ln \frac{\mu_0}{\mu}\right)$$

  - *shifts* logarithms $\ln(k/\mu_0)$ into $\ln(k/\mu)$
  - sufficient if $F(k, \mu_0)$ is known “exactly” at reference scale μ_0
    Example: PDF evolution

- Purpose of resummation:
  - *predict* all logarithms of $F(k, \mu)$
  - such that $F(k, \mu_0)$ is free of large logs $\ln(k/\mu_0)$
    $\to F(k, \mu_0)$ can be calculated perturbatively

Toy example.

Simple toy function:

- Toy function (neglecting α_s running)

$$F(k,\mu) = F(k,\mu_0)\exp\left(\alpha_s \ln\frac{\mu_0}{\mu}\right)$$

- $\mu_0 = k$ eliminates all logarithms $\ln(k/\mu_0)$ in boundary term

$$F(k,\mu_0 = k) = 1 + \cdots$$

- All logarithms are now fully resummed:

$$F(k,\mu) = (1+\cdots)\exp\left(\alpha_s \ln\frac{k}{\mu}\right)$$
$$= 1 + \alpha_s \ln\frac{k}{\mu} + \frac{1}{2}\alpha_s^2 \ln^2\frac{k}{\mu^2} + \cdots$$

- All-order structure of $F(k,\mu)$ easily derived from evolution equation

Toy example in distribution space.

Plus distributions:

- For many observables: $F(k,\mu)$ contains plus distributions (Encodes the cancellation of IR divergences $k \to 0$)

$$\Big[\theta(k) g(k,\mu)\Big]_+^\mu = \theta(k) g(k,\mu) \qquad \text{for } k \neq 0$$

$$\int^\mu \mathrm{d}k \Big[\theta(k) g(k,\mu)\Big]_+^\mu = 0$$

- Important type:

$$\mathcal{L}_n(k,\mu) = \left[\frac{\theta(k)}{k} \ln^n \frac{k}{\mu}\right]_+^\mu$$

Distributional toy example:

$$F(k,\mu) = \delta(k) + \alpha_s \mathcal{L}_0(k,\mu) + \alpha_s^2 \mathcal{L}_1(k,\mu) + \cdots$$

- fulfills the RGE

$$\mu \frac{\mathrm{d}F(k,\mu)}{\mathrm{d}\mu} = -\alpha_s F(k,\mu)$$

Toy example in distribution space.

- How to derive the toy distribution

$$F(k,\mu) = \delta(k) + \alpha_s \mathcal{L}_0(k,\mu) + \alpha_s^2 \mathcal{L}_1(k,\mu) + \cdots$$

- from its RGE? $$\mu \frac{\mathrm{d}F(k,\mu)}{\mathrm{d}\mu} = -\alpha_s F(k,\mu)$$

- Formal solution: $$F(k,\mu) = F(k,\mu_0) \exp\left(\alpha_s \ln \frac{\mu_0}{\mu}\right)$$

- Need to mimic normal scale setting:
  - Minimize boundary term

$$F(k, \mu_0 = k|_+) = \delta(k) + \cdots$$

  - Predict higher logarithmic distributions

$$\delta(k) \exp\left(\alpha_s \ln \frac{\mu_0}{\mu}\right)\bigg|_{\mu_0 = k|_+} = \delta(k) + \alpha_s \mathcal{L}_0(k,\mu) + \cdots$$

Distributional scale setting.

Definition:

$$D(k, \mu = k|_+) \equiv \frac{\mathrm{d}}{\mathrm{d}k}\left[\int^k \mathrm{d}k'\, D(k', \mu = k)\right]$$

- Setting $\mu = k$ inside integral well-defined
- No effect for arbitrary $\mu = \mu_0$: $D(k, \mu = \mu_0|_+) = D(k, \mu = \mu_0)$

Illustration:

$$\begin{aligned}\mathcal{L}_n(k, \mu = k|_+) &= \frac{\mathrm{d}}{\mathrm{d}k}\int^k \mathrm{d}k' \left[\frac{\theta(k')}{k'} \ln^n \frac{k'}{\mu}\right]_+^\mu \Bigg|_{\mu=k} \\ &= \frac{\mathrm{d}}{\mathrm{d}k}\left[\frac{\theta(k)}{n+1} \ln^{n+1} \frac{k}{\mu}\right]_{\mu=k} \\ &= \frac{\mathrm{d}}{\mathrm{d}k} 0 = 0\end{aligned}$$

- Minimizes distributions $\mathcal{L}_n(k, \mu)$ like ordinary logarithms $\ln^n(k/\mu)$ ✓

Distributional scale setting.

Definition:

$$D(k, \mu = k|_+) \equiv \frac{\mathrm{d}}{\mathrm{d}k}\left[\int^k \mathrm{d}k'\, D(k', \mu = k)\right]$$

- Setting $\mu = k$ inside integral well-defined
- No effect for arbitrary $\mu = \mu_0$: $D(k, \mu = \mu_0|_+) = D(k, \mu = \mu_0)$

Properties: $(n, m \geq 0)$

$$\delta(k)\, \ln^{n+1} \frac{\mu_0}{\mu}\bigg|_{\mu_0 = k|_+} = (n+1)\left[\frac{\theta(k)}{k} \ln^n \frac{k}{\mu}\right]_+^\mu$$

$$(m+1)\mathcal{L}_m(k, \mu)\, \ln^n \frac{\mu_0}{\mu}\bigg|_{\mu_0 = k|_+} = (n+m+1)\mathcal{L}_{m+n}(k, \mu)$$

- Distributions are essentially treated like ordinary logarithms

Illustration: The distributional exponential.

- Formal solution: (neglecting α_s running)

$$F(k,\mu) = F(k,\mu_0)\exp\left(\alpha_s \ln\frac{\mu_0}{\mu}\right)$$

- Correct minimization of distributional boundary term: $\mu_0 = k|_+$:

$$\begin{aligned}
F(k,\mu) &= \frac{\mathrm{d}}{\mathrm{d}k}\int^k \mathrm{d}k'\, F(k',\mu_0)\exp\left(\alpha_s \ln\frac{\mu_0}{\mu}\right)\bigg|_{\mu_0=k} \\
&= \frac{\mathrm{d}}{\mathrm{d}k}\int^k \mathrm{d}k'\, F(k',k)\exp\left(\alpha_s \ln\frac{k}{\mu}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}k}\theta(k)(1+\cdots)\exp\left(\alpha_s \ln\frac{k}{\mu}\right) \\
&= \delta(k) + \alpha_s\left[\frac{\theta(k)}{k}\exp\left(\alpha_s \ln\frac{k}{\mu}\right)\right]_+^{\mu} \\
&= \delta(k) + \alpha_s \mathcal{L}_0(k,\mu) + \alpha_s^2 \mathcal{L}_1(k,\mu) + \cdots
\end{aligned}$$

- $\mu_0 = k|_+$ correctly produces the distributional exponential ✓

Alternative solution of toy function.

- Toy RGE $$\mu \frac{\mathrm{d}F(k,\mu)}{\mathrm{d}\mu} = -\alpha_s F(k,\mu)$$
- Ansatz $$F(k,\mu) = f_0(\mu)\delta(k) + \big[\theta(k) f_1(k,\mu)\big]_+^\mu$$
- Apply derivative:
$$\mu \frac{\mathrm{d}F(k,\mu)}{\mathrm{d}\mu} = \delta(k)\left(\mu \frac{\mathrm{d}f_0(\mu)}{\mathrm{d}\mu} - \mu f_1(\mu,\mu)\right) + \left[\theta(k)\mu \frac{\mathrm{d}f_1(k,\mu)}{\mathrm{d}\mu}\right]_+^\mu$$
$$\stackrel{!}{=} -\alpha_s f_0(\mu)\delta(k) - \alpha_s \big[\theta(k) f_1(k,\mu)\big]_+^\mu$$
- Coupled system of RGEs:
$$\mu \frac{\mathrm{d}f_1(k,\mu)}{\mathrm{d}\mu} = -\alpha_s f_1(k,\mu)\,,$$
$$\mu \frac{\mathrm{d}f_0(\mu)}{\mathrm{d}\mu} = -\alpha_s f_0(\mu) + \mu f_1(\mu,\mu)\,.$$
- $f_0(\mu)$ is pure boundary, determines $f_1(k,\mu)$:
$$f_1(k,\mu) = \frac{\mathrm{d}}{\mathrm{d}k} f_0(k) \exp\left(\alpha_s \ln \frac{k}{\mu}\right).$$

Alternative solution of toy function.

- Toy RGE
$$\mu \frac{\mathrm{d}F(k,\mu)}{\mathrm{d}\mu} = -\alpha_s F(k,\mu)$$
- Ansatz
$$F(k,\mu) = f_0(\mu)\delta(k) + \big[\theta(k) f_1(k,\mu)\big]_+^\mu$$
- Simplest boundary: $f_0(\mu) = 1$

$$F(k,\mu) = \delta(k) + \left[\theta(k)\frac{\mathrm{d}}{\mathrm{d}k}\exp\left(\alpha_s \ln\frac{k}{\mu}\right)\right]_+^\mu$$
$$= \delta(k) - \alpha_s\left[\theta(k)\exp\left(\alpha_s \ln\frac{k}{\mu}\right)\right]_+^\mu$$

- Exactly matches the solution using $\mu_0 = k|_+$
- Illustrates equivalence of both techniques

Summary: distributional scale setting.

- Minimizing distributional logarithms requires distributional scale setting

$$D(k, \mu = k|_+) \equiv \frac{\mathrm{d}}{\mathrm{d}k}\left[\int^k \mathrm{d}k'\, D(k', \mu = k)\right]$$

- Two-dimensional generalization:

$$D(\vec{p}_T, \mu = p_T|_+) \equiv \frac{1}{2\pi p_T}\frac{\mathrm{d}}{\mathrm{d}p_T}\left[\int_{|\vec{k}_T| \leq p_T} \mathrm{d}^2\vec{k}_T\, D(\vec{k}_T, \mu = p_T)\right]$$

- Treats distributional logarithms essentially like ordinary logarithms

- Allows to solve RGEs (i.e. resummation) directly in distribution space